

On chaotic minimal center of attraction of a Lagrange stable motion for topological semi flows

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Abstract

Let $f: \mathbb{R}_+ \times X \rightarrow X$ be a topological semi flow on a Polish space X . In 1977, Karl Sigmund conjectured that if there is a point x in X such that the motion $f(t, x)$ has just X as its minimal center of attraction, then the set of all such x is *residual* in X . In this paper, we present a positive solution to this conjecture and apply it to the study of chaotic dynamics of minimal center of attraction of a motion.

Keywords: Minimal center of attraction · Chaotic motion · Semi flow

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1. Introduction

By a C^0 -semi flow over a metric space X , we here mean a transformation $f: \mathbb{R}_+ \times X \rightarrow X$ where $\mathbb{R}_+ = [0, \infty)$, which satisfies the following three conditions:

- (1) The initial condition: $f(0, x) = x$ for all $x \in X$.
- (2) The condition of continuity: if there be given two convergent sequences $t_n \rightarrow t_0$ in \mathbb{R}_+ and $x_n \rightarrow x_0$ in X , then $f(t_n, x_n) \rightarrow f(t_0, x_0)$ as $n \rightarrow \infty$.
- (3) The semigroup condition: $f(t_2, f(t_1, x)) = f(t_1 + t_2, x)$ for any x in X and any times t_1, t_2 in \mathbb{R}_+ .

Sometimes we write $f(t, x) = f^t(x)$ for any $t \geq 0$ and $x \in X$; and for any given point $x \in X$ we call $f(t, x)$ a motion in X and $O_f(x) = f(\mathbb{R}_+, x)$ the orbit starting from the point x . If $O_f(x)$ is precompact (i.e. $\overline{O_f(x)}$ is compact) in X , then we say that $f(t, x)$ is *Lagrange stable*.

We refer to any subset Λ of X as an *invariant* set if $f(t, x) \in \Lambda$ for each point $x \in \Lambda$ and any time $t \geq 0$. In dynamical systems, statistical mechanics and ergodic theory, we shall have to do with “probability of sojourn of a motion $f(t, x)$ in a given region E of X ” as $t \rightarrow +\infty$:

$$P(f(t, x) \in E) = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \mathbb{1}_E(f(t, x)) dt,$$

where $\mathbb{1}_E(x)$ is the characteristic function of the set E on X .

This motivates H.F. Hilmy to introduce following important concept, which was discussed in [13, 15, 16, 12], for example.

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Definition 1.1 (Hilmy 1936 [10]). Given any $x \in X$, a closed subset C_x of X is called the *center of attraction of the motion* $f(t, x)$ as $t \rightarrow +\infty$ if $P(f(t, x) \in B_\varepsilon(C_x)) = 1$ for all $\varepsilon > 0$, where $B_\varepsilon(C_x)$ denotes the ε -neighborhood around C_x in X . If the set C_x does not admit a proper subset which is likewise a center of attraction of the motion $f(t, x)$ as $t \rightarrow +\infty$, then C_x is called the *minimal center of attraction of the motion* $f(t, x)$ as $t \rightarrow +\infty$.

First of all, by the classical Cantor-Baire theorem and Zorn's lemma we can obtain the following basic existence lemma.

Lemma 1.2. *Let $f: \mathbb{R} \times X \rightarrow X$ be a C^0 -semi flow on a metric space X . Then each Lagrange stable motion $f(t, x)$ always possesses the minimal center of attraction.*

From now on, by C_x we will understand the minimal center of attraction of a Lagrange stable motion $f(t, x)$ as $t \rightarrow +\infty$. In [16], Karl Sigmund gave an intrinsic characterization for C_x . In this paper, we shall study the generic property and chaotic dynamics occurring in C_x for a Lagrange stable motion $f(t, x)$.

Just as the existence of one point which is topologically transitive implies that a residual set is topologically transitive, in 1977 Karl Sigmund raised the following open problem:

Conjecture 1.3 (K. Sigmund 1977 [16, Remark 4]). For any homeomorphism f of a compact metric space X , the set of points $x \in X$ with $C_x = X$, if nonempty, is residual in X .

Since a residual set contains a dense and G_δ subset of X , it is very large from the viewpoint of topology. Although Sigmund's conjecture is of interest, there has not been any progresses on it since 1977 except f satisfies the "specification" property ([16, Proposition 6]). In Section 2, we will present a positive solution to Sigmund's conjecture without any imposed shadowing assumption like specification, which is stated as follows:

Theorem 1.4. *Let $f: \mathbb{R}_+ \times X \rightarrow X$ be a C^0 -semi flow on a Polish space X . If some motion $f(t, x)$ is such that $C_x = X$, then the set $\{x \in X: C_x = X\}$ is residual in X .*

Our argument of Theorem 1.4 below also works for discrete-time case. Thus we can obtain the following:

Corollary. *For any continuous transformation f of a Polish space X , the set of points $x \in X$ with $C_x = X$, if nonempty, is residual in X .*

We now turn to some applications of Theorem 1.4. For our convenience, we first introduce two notions for a C^0 -semi flow $f: \mathbb{R}_+ \times X \rightarrow X$ on a Polish space X .

Definition 1.5. An f -invariant subset Λ of X is referred to as *generic* if there exists some point $x \in \Lambda$ with $\Lambda = C_x$.

According to Conjecture 1.3 (or precisely speaking Theorem 1.4) for any generic minimal center of attraction C_x of a motion $f(t, x)$, the set of points $y \in C_x$ with $C_y = C_x$ is residual in C_x .

Definition 1.6. We say that a motion $f(t, x)$ is *chaotic* for f if there can be found some point $y \in X$ such that

$$\liminf_{t \rightarrow +\infty} d(f(t, x), y) = 0, \quad \limsup_{t \rightarrow +\infty} d(f(t, x), y) > 0$$

and

$$\liminf_{t \rightarrow +\infty} d(f(t, x), f(t, y)) = 0, \quad \limsup_{t \rightarrow +\infty} d(f(t, x), f(t, y)) > 0.$$

By using Theorem 1.4, we will show that if C_x is not generic, then the chaotic behavior occurs near C_x ; see Theorems 3.1 and 3.2 stated and proved in Section 3. On the other hand whenever C_x is generic and it is not “very simple”, then chaotic motions are generic in C_x ; that is the following

Theorem 1.7. *Let $f(t, p)$ be a Lagrange stable motion in a Polish space X . If C_p is generic and itself is not a minimal subset of (X, f) , then there exists a residual subset S of C_p such that $f(t, x)$ is chaotic for each $x \in S$.*

In Section 4 we will consider a relationship of the minimal center of attraction of a motion with the pointwise recurrence.

Finally we shall consider the multiply attracting of the minimal center of attraction of a motion in Section 5.

2. Proof of Sigmund’s conjecture

This section will be devoted to proving Karl Sigmund’s Conjecture 1.3 in the continuous-time case; that is, Theorem 1.4.

Recall that if a continuous surjective map $T: X \rightarrow X$ is topologically transitive, then for a countable basis $U_1, U_2, \dots, U_n, \dots$ of the underlying space X , the set

$$\{x \in X \mid \overline{O_T(x)} = X\} = \bigcap_{n=1}^{+\infty} \bigcup_{m=0}^{+\infty} T^{-m}(U_n), \quad \text{where } O_T(x) = \{T^n x : n \in \mathbb{Z}_+\},$$

is a dense G_δ set in X because $\bigcup_{m=0}^{+\infty} T^{-m}(U_n)$ is open and dense in X . It is easy to see that this standard argument for topological transitivity does not work here for Sigmund’s conjecture (or Theorem 1.4). So we need a new idea that will explore the times of sojourn of a motion in a domain.

Given any $x \in X$, let \mathcal{U}_x be the neighborhood system of the point x in X . To prove Conjecture 1.3, we will need a classical result, which shows that C_x consists of the points $y \in X$ which are interesting for x .

Lemma 2.1 ([10, 15] for C^0 -flow). *Let $f: \mathbb{R}_+ \times X \rightarrow X$ be a C^0 -semi flow on a metric space X . Then for any $x \in X$, there holds*

$$C_x = \left\{ y \in X \mid \limsup_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \mathbb{1}_U(f(t, x)) dt > 0 \ \forall U \in \mathcal{U}_y \right\}.$$

Proof. For self-closeness, we present an independent proof for this lemma which is shorter than that of [15]. Let $x \in X$ and write

$$I(x) = \left\{ y \in X \mid \limsup_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \mathbb{1}_U(f(t, x)) dt > 0 \ \forall U \in \mathcal{U}_y \right\}.$$

We first claim that $C_x \subseteq I(x)$. Indeed, for any $q \in C_x$, let $U \in \mathcal{U}_q$ be a neighborhood of q in X ; then

$$\limsup_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \mathbb{1}_U(f(t, x)) dt > 0.$$

Otherwise, one would find some $\varepsilon > 0$ so that

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \mathbb{1}_{B_{3\varepsilon}(q)}(f(t, x)) dt = 0.$$

Further $C_x - B_{2\varepsilon}(q)$ is a center of attraction of the motion $f(t, x)$ and we thus arrive at a contradiction to the minimality of C_x .

Finally we assert that $C_x \supseteq I(x)$. By contradiction, let $q \in I(x) - C_x$ and then we can take some $\varepsilon > 0$ such that $d(q, C_x) \geq 3\varepsilon$. Set

$$N(B_\varepsilon(q)) = \{t \geq 0 \mid f(t, x) \in B_\varepsilon(q) \text{ for } 1 \leq i \leq l\}$$

and

$$N(B_\varepsilon(C_x)) = \{t \geq 0 \mid f(t, x) \in B_\varepsilon(C_x) \text{ for } 1 \leq i \leq l\}.$$

Clearly $N(B_\varepsilon(q)) \cap N(B_\varepsilon(C_x)) = \emptyset$. However, by definitions, $N(B_\varepsilon(q))$ has positive upper density and $N(B_\varepsilon(C_x))$ has density 1 in $[0, \infty)$. This is a contradiction.

The proof of Lemma 2.1 is thus completed. \square

In Definition 1.1 we do not require the f -invariance of C_x . However, it is actually f -invariant by Lemma 2.1.

Corollary 2.2. *Let $f: \mathbb{R}_+ \times X \rightarrow X$ be a C^0 -semi flow on a metric space X . For any $x \in X$, C_x is f -invariant.*

Since for any real number $\theta > 0$ and any integer $N \geq 0$ there holds

$$0 \leq \int_0^\theta \mathbb{1}_U(f(N\theta + t, x)) dt \leq \theta,$$

then Lemma 2.1 implies immediately the following.

Corollary 2.3. *Let $f: \mathbb{R}_+ \times X \rightarrow X$ be a C^0 -semi flow on a Polish space X . For any $x \in X$ and any $\theta > 0$, there holds*

$$C_x = \left\{ y \in X \mid \limsup_{N \ni N \rightarrow +\infty} \frac{1}{N} \int_0^{N\theta} \mathbb{1}_U(f(t, x)) dt > 0 \ \forall U \in \mathcal{U}_y \right\}.$$

Here $\mathbb{N} = \{1, 2, \dots\}$.

Proof. The statement follows from that $T = N_T\theta + r_T$ where $N_T \in \mathbb{N}$, $0 \leq r_T < \theta$ for any $T \geq 1$ and that

$$\limsup_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \mathbb{1}_U(f(t, x)) dt = \limsup_{N \ni N \rightarrow +\infty} \frac{1}{N\theta} \int_0^{N\theta} \mathbb{1}_U(f(t, x)) dt$$

for any $U \in \mathcal{U}_y$ and any $y \in X$. \square

We are now ready to prove one of our main statements—Theorem 1.4—by applying Lemma 2.1 and Corollaries 2.2 and 2.3.

Proof of Theorem 1.4. Write $\Theta = \{x \in X : C_x = X\}$ and let $x \in X$ be such that $C_x = X$. Then from Lemma 2.1, it follows that the orbit $O_f(x) = f(\mathbb{R}_+, x)$ is dense in X and that $O_f(x) \subseteq \Theta$ such that for any $y \in O_f(x)$,

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \int_0^N \mathbb{1}_U(f(t, x)) dt = \limsup_{N \rightarrow \infty} \frac{1}{N} \int_0^N \mathbb{1}_U(f(t, y)) dt,$$

for every nonempty $U \in \mathcal{T}_X$, where \mathcal{T}_X is the topology of the space X .

Let $\mathcal{U} = \{U_i\}_{i=1}^\infty$ be an any given countable base of the topology \mathcal{T}_X of the state space X . Then by Corollary 2.3, we can choose a sequence of positive integers $L_i, i = 1, 2, \dots$, such that

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \int_0^N \mathbb{1}_{U_i}(f(t, x)) dt > \frac{1}{L_i},$$

for all $i = 1, 2, \dots$. For each i , write

$$\Theta_i = \left\{ y \in X \mid \forall n_0 \in \mathbb{N}, \exists n > n_0 \text{ with } \int_0^{nL_i} \mathbb{1}_{U_i}(f(t, y)) dt > n \right\}.$$

From the continuity of $f(t, y)$ with respect to (t, y) and

$$\Theta_i = \bigcap_{n_0=1}^\infty \bigcup_{n > n_0} \left\{ y \in X \mid \int_0^{nL_i} \mathbb{1}_{U_i}(f(t, y)) dt > n \right\},$$

it follows that Θ_i is a G_δ subset of X . Because x belongs to Θ_i by noting $N = n_N L_i + r_N$ and $\lim_{N \rightarrow \infty} \frac{r_N}{N} = 0$ where $0 \leq r_N < L_i$ and then similarly $O_f(x) \subseteq \Theta_i$, there follows $\bigcap_{i=1}^\infty \Theta_i$ is a dense and G_δ set in X .

Since for any $y \in \bigcap_{i=1}^\infty \Theta_i$ we have

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \int_0^N \mathbb{1}_{U_i}(f(t, y)) dt \geq \frac{1}{L_i} > 0$$

for any $U_i \in \mathcal{U}$ and \mathcal{U} is a base of the topology \mathcal{T}_X of the space X , we see that $y \in \Theta$ from Corollary 2.3. Therefore, Θ is a residual set in X .

This completes the proof of Theorem 1.4. \square

Finally we mention that the statement of Theorem 1.4 is also valid for a continuous transformation $f: X \rightarrow X$ of a Polish space X .

3. Li-Yorke chaotic pairs and sensitive dependence on initial data

In this section, we shall apply Theorem 1.4 to the study of chaos of a topological dynamical system on a Polish space X .

Let $f: \mathbb{R}_+ \times X \rightarrow X$ be a C^0 -semi flow on the Polish space (X, d) . Recall that two points $x, y \in X$ is called a *Li-Yorke chaotic pair* for f if

$$\limsup_{t \rightarrow +\infty} d(f(t, x), f(t, y)) > 0 \quad \text{and} \quad \liminf_{t \rightarrow +\infty} d(f(t, x), f(t, y)) = 0.$$

That is to say, x is proximal to y but not asymptotical. If there can be found an uncountable set $S \subset X$ such that every pair of points $x, y \in S, x \neq y$, is a Li-Yorke chaotic pair for f , then we say f is *Li-Yorke chaotic*; see, e.g., Li and Yorke 1975 [14].

3.1. Nongeneric case

Let $f: \mathbb{R}_+ \times X \rightarrow X$ be a C^0 -semi flow on a compact metric space. The following theorem shows that if C_x has no the generic dynamics in the sense of Definition 1.5, then f has the chaotic dynamics.

Theorem 3.1. *Given any $x \in X$, if C_x is not generic, then one can find some point $q \in \Delta$, for any closed f -invariant subset $\Delta \subseteq C_x$, such that (x, q) is a Li-Yorke chaotic pair for f .*

Proof. Given any $x \in X$, let C_x be not generic in the sense of Definition 1.5. Let Δ be an f -invariant nonempty closed subset of C_x . Then by Theorem 1.4 it follows that $x \notin C_x$. Moreover from Definition 1.1, we can obtain that x is proximal to Δ ; that is,

$$\liminf_{t \rightarrow +\infty} d(f(t, x), f(t, \Delta)) = 0.$$

Then from [1, 6] also [7, Proposition 8.6], it follows that there exists some point $q \in \Delta$ such that

$$\liminf_{t \rightarrow +\infty} d(f(t, x), f(t, q)) = 0.$$

We claim that $\limsup_{t \rightarrow +\infty} d(f(t, x), f(t, q)) > 0$. Otherwise, $\lim_{t \rightarrow +\infty} d(f(t, x), f(t, q)) = 0$ and thus $C_x = C_q$; and then C_x is generic by Theorem 1.4.

The proof of Theorem 3.1 is therefore complete. \square

Corollary 1. *Given any $x \in X$, if C_x is not generic, then one can find some point $q \in C_x$ such that (x, q) is a Li-Yorke chaotic pair for f and the set*

$$N_f(x, B_\varepsilon(q)) = \{t \geq 0: f(t, x) \in B_\varepsilon(q)\}$$

is a central set in \mathbb{R}_+ , which has positive upper density.

Proof. Let $\Delta \subset C_x$ be a minimal set. Then there can be found by Theorem 3.1 a point $q \in \Delta$ such that (x, q) is a Li-Yorke chaotic pair for f . By definition (cf. [7, Definition 8.3] and [5, Definition 7.2]) $N_f(x, B_\varepsilon(q))$ is a central set of \mathbb{R}_+ for each $\varepsilon > 0$. In addition, by Lemma 2.1 it follows that $N_f(x, B_\varepsilon(q))$ has positive upper density. This proves the corollary. \square

Corollary 2. *Let there exist a fixed point or a periodic orbit in the minimal center C_x of attraction of a motion $f(t, x)$. Then we can find a Li-Yorke chaotic pair near C_x .*

Proof. First if C_x is generic in the sense of Definition 1.5, then f restricted to it is topologically transitive and has a fixed point or a periodic orbit. Then by Huang and Ye 2002 [11, Theorem 4.1], it follows that f restricted to C_x is chaotic in the sense of Li and Yorke.

On the other hand, if C_x is not generic in the sense of Definition 1.5, then from Theorem 3.1 it follows that there always exists a Li-Yorke chaotic pair near C_x .

The proof of the corollary is thus complete. \square

Recall that a motion $f(t, x)$ is referred to as a *Birkhoff recurrent motion* of f if $\overline{O_f(x)}$ is minimal (cf. [15, 3]). It is also called “uniformly recurrent” in [7] and “almost periodic” of von Neumann in [9] in the discrete-time case.

Motivated by [2, 8, 4] we can obtain the following theorem on sensitivity on initial data near the minimal center of attraction of a motion.

Theorem 3.2. *Let C_x , for a motion $f(t, x)$, be not generic. If the Birkhoff recurrent points of f are dense in C_x , then f has the sensitive dependence on initial data near C_x in the sense that one can find a sensitive constant $\epsilon > 0$ such that for any $a \in X$, $\hat{x} \in C_x$ and any $U \in \mathcal{U}_{\hat{x}}$, there exists some point $c \in U$ with $\limsup_{t \rightarrow +\infty} d(f(t, a), f(t, c)) \geq \epsilon$.*

Proof. Since C_x is not generic, by Theorem 1.4 it follows that it is not minimal and so it contains at least two different motions of f far away each other. Thus one can find a number $\delta > 0$ such that for all $\hat{x} \in C_x$ there exists a corresponding motion $f(t, q)$ in C_x , not necessarily recurrent, such that

$$d(\hat{x}, \overline{O_f(q)}) \geq \delta,$$

where $d(\hat{x}, A) = \inf_{a \in A} d(\hat{x}, a)$ for any subset A of X . We will show that f has sensitive dependence on initial data with sensitivity constant $\epsilon = \delta/4$ following the idea of [4, Theorem 4].

For this, we let \hat{x} be an arbitrary point in C_x and let U be an arbitrary neighborhood of \hat{x} in X . Since the Birkhoff recurrent motions of (X, f) are dense in C_x from assumption of the theorem, there exists a Birkhoff recurrent point $p \in U \cap B_{\epsilon/2}(\hat{x}) \cap C_x$, where $B_r(\hat{x})$ is the open ball of radius r centered at \hat{x} in X . As we noted above, there must exist another point $q \in C_x$ whose orbit $O_f(q)$ is of distance at least 4ϵ from the given point \hat{x} .

Let $\eta > 0$ be such that $\eta < \epsilon/2$. Then from the Birkhoff recurrence of the motion $f(t, p)$, it follows that one can find a constant $T = T(\eta, p) > 0$ such that for any $\gamma \geq 0$, there is some moment $t_\gamma \in [\gamma, \gamma + T)$ verifying that

$$d(p, f^{t_\gamma}(p)) < \eta.$$

For the given q , we simply write

$$V = \bigcap_{t \in [0, 2T)} f^{-t}(B_\epsilon(f^t(q))), \quad \text{where } f^{-t}(\cdot) = f(t, \cdot)^{-1}.$$

Clearly from the continuity of topological flow, it follows that V is a neighborhood of q in X but not necessarily open, and it is nonempty since $q \in V$.

Since C_x is the minimal center of attraction of the motion $f(t, x)$, from Lemma 2.1 it follows that there is at least one point $z \in U \cap B_\epsilon(\hat{x})$ such that $f^N(z) \in V$ for some sufficiently large number $N \gg T$. Let

$$N = jT - r \quad \text{where } 0 \leq r < T, \quad j \in \mathbb{N},$$

and

$$t_{jT} \in [jT, (j+1)T) \quad \text{such that } d(p, f^{t_{jT}}(p)) < \eta.$$

Then $0 \leq t_{jT} - N < 2T$.

By construction, one has

$$f^{t_{jT}}(z) = f^{t_{jT}-N}(f^N(z)) \in f^{t_{jT}-N}(V) \subseteq B_\epsilon(f^{t_{jT}-N}(q)).$$

From the triangle inequality of metric, it follows that

$$\begin{aligned} d(f^{t_{jT}}(p), f^{t_{jT}}(z)) &\geq d(p, f^{t_{jT}}(z)) - \eta \\ &\geq d(\hat{x}, f^{t_{jT}}(z)) - d(p, \hat{x}) - \eta \\ &\geq d(\hat{x}, f^{t_{jT}-N}(q)) - d(f^{t_{jT}-N}(q), f^{t_{jT}}(z)) - d(p, \hat{x}) - \eta. \end{aligned}$$

Consequently, since $\eta < \epsilon/2$, $p \in B_{\epsilon/2}(\hat{x})$ and $f^{t_{jT}}(z) \in B_{\epsilon}(f^{t_{jT}-N}(q))$, it holds that

$$d(f^{t_{jT}}(p), f^{t_{jT}}(z)) > 2\epsilon.$$

Therefore from the triangle inequality again, one can obtain either

$$d(f^{t_{jT}}(\hat{x}), f^{t_{jT}}(z)) > \epsilon$$

or

$$d(f^{t_{jT}}(\hat{x}), f^{t_{jT}}(p)) > \epsilon.$$

Repeating this argument for another likewise N bigger than $(j+2)T$, one can find a sequence $t_n = j_n T \uparrow +\infty$ as $n \rightarrow +\infty$ such that either

$$d(f^{t_n}(\hat{x}), f^{t_n}(z)) > \epsilon$$

or

$$d(f^{t_n}(\hat{x}), f^{t_n}(p)) > \epsilon,$$

for all $n \geq 1$. Thus in either case, we have found a point $\hat{y} \in U$ such that

$$\limsup_{t \rightarrow +\infty} d(f^t(\hat{x}), f^t(\hat{y})) \geq \epsilon.$$

Now for any $a \in X$, using the triangle inequality once more, we see either

$$\limsup_{t \rightarrow +\infty} d(f^t(\hat{x}), f^t(a)) \geq \frac{\epsilon}{3}$$

or

$$\limsup_{t \rightarrow +\infty} d(f^t(\hat{y}), f^t(a)) \geq \frac{\epsilon}{3}.$$

Since \hat{x}, U both are arbitrary and $\hat{y} \in U$, hence the proof of Theorem 3.2 is complete. \square

We note that if (C_x, f) is distal (cf. [7, Definition 8.2]) and not minimal and even not topologically transitive, then the conditions of Theorem 3.2 hold; i.e., the Birkhoff recurrent points are dense in C_x . In fact, C_x consists of Birkhoff recurrent points ([7, Corollary of Theorem 8.7]) and it is not topologically transitive.

3.2. Generic case

Let $f: \mathbb{R}_+ \times X \rightarrow X$ be a C^0 -semi flow on a Polish space and $f(t, p)$ a Lagrange stable motion. Then the minimal center C_p of attraction of the motion $f(t, p)$ is always existent. Theorem 1.7 is just a corollary of the following

Theorem 3.3. *Let C_p be generic and not a minimal subset of (X, f) . Then there exists a residual subset S of C_p such that for any $x \in S$ and any minimal subset $\Lambda \subset C_p$, there corresponds some point $y \in \Lambda$ with the properties: x, y form a Li-Yorke chaotic pair for f and*

$$\liminf_{t \rightarrow +\infty} d(f(t, x), y) = 0 \quad \text{and} \quad \limsup_{t \rightarrow +\infty} d(f(t, x), y) \geq \frac{1}{2} \text{diam}(C_p).$$

Proof. Since C_p is generic in the sense of Definition 1.5, there exists some point $q \in C_p$ with $C_q = C_p$. Therefore by Theorem 1.4, there is a residual subset S of C_p such that $C_x = C_p$ for all point x in S . Because C_p is not a minimal subset of X by hypothesis of Theorem 1.7, $\overline{O_f(x)}$ is not minimal for each $x \in S$.

Let Λ be a minimal subset of C_p . Then each $x \in S$ is proximal to Λ . Moreover by [7, Theorem 8.7], it follows that for every $x \in S$, there corresponds some point $y \in \Lambda$ such that x is proximal to y and $f(t, y)$ is Birkhoff recurrent (or uniformly recurrent). Clearly, x and y is a Li-Yorke chaotic pair for f . In addition, from Lemma 2.1 follows that

$$\limsup_{t \rightarrow +\infty} d(f(t, x), y) \geq \frac{1}{2} \text{diam}(C_p)$$

and

$$\liminf_{t \rightarrow +\infty} d(f(t, x), y) = 0.$$

This completes the proof of Theorem 3.3. \square

Therefore by Theorems 3.1 and 3.3, it follows that every Lagrange stable motion $f(t, x)$ is chaotic in the sense of Definition 1.6 if its minimal center C_x of attraction is not a minimal subset of (X, f) .

4. Quasi-weakly almost periodic motion

In this section, we let $f: \mathbb{R}_+ \times X \rightarrow X$ be a C^0 -semi flow on the compact metric space X .

Definition 4.1 (Huang-Zhou 2012 [12]). A point x in X is called a *quasi-weakly almost periodic point* of f if for any $\varepsilon > 0$ there exists an integer $N = N(\varepsilon) \geq 1$ and an increasing positive integer sequence $\{n_j\}$ with the property that for each j there are $0 = t_0 < t_1 < \dots < t_{n_j} < n_j N$ with $t_{i+1} - t_i \geq 1$ such that $f(t_i, x) \in B_\varepsilon(x)$ for all $i = 1, \dots, n_j$.

As results of the statements of Lemma 2.1 and Theorem 1.4, we can obtain the following two results.

Proposition 4.2. The following statements are equivalent to each other.

- (1) $x \in X$ is a quasi-weakly almost periodic point of f .
- (2) $x \in C_x$.

Proof. (1) \Rightarrow (2) follows from Lemma 2.1. (2) \Rightarrow (1) follows from Lemma 2.1 and the local section theorem of Bebutov (cf. [15, Theorem V.2.14]). \square

Proposition 4.2 has been proved in [12] in the case where x is a Poisson stable point of f , i.e., there is a sequence $t_n \rightarrow \infty$ such that $f(t_n, x) \rightarrow x$ as $n \rightarrow \infty$.

Proposition 4.3. If $x \in C_x$, then the set $\{y \in C_x \mid y \in C_y = C_x\}$ is dense and G_δ relative to the subspace C_x .

Proof. The statement follows from Theorem 1.4 with C_x replacing of X . \square

5. Minimal center of multi-attraction of a motion

Let $f: \mathbb{R}_+ \times X \rightarrow X$ be a C^0 -semi flow on a Polish space X . From now on, by $\lambda(dt)$ we denote the standard Haar (Lebesgue) measure on \mathbb{R} . We will need the following simple but useful fact.

Lemma 5.1. *Let S be a measurable subset of \mathbb{R}_+ and $\tau > 0$. If S has the density α , i.e.,*

$$D(S) := \lim_{T \rightarrow +\infty} \frac{\lambda(S \cap [0, T])}{T} = \alpha,$$

then $\tau S = \{\tau t: t \in S\}$ also has the density α in \mathbb{R}_+ .

Proof. Let $\tau > 0$ be any given. Since $\lambda(\tau A) = \tau \lambda(A)$ and $\lambda(\tau S \cap [0, T]) = \tau \lambda(S \cap [0, T\tau^{-1}])$, hence it follows that $D(\tau S) = 1$. This proves the lemma. \square

It should be noted here that there is no an analogous result for the discrete-time \mathbb{Z}_+ ; for example, $S = \{0, 2, 4, 6, \dots\}$ has the density $\frac{1}{2}$ but $\frac{1}{2}S$ has the density 1 in \mathbb{Z}_+ .

The following lemma shows that every minimal center of attraction of a motion $f(t, x)$ is multiply attracting as $t \rightarrow +\infty$.

Lemma 5.2. *Let C_x be existent for a motion $f(t, x)$ as $t \rightarrow +\infty$. Then for any $t_1 > 0, \dots, t_l > 0$ where $l \in \mathbb{N}$ and any $\varepsilon > 0$,*

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \mathbb{1}_{B_\varepsilon(C_x)}(f(t_1 t, x)) \cdots \mathbb{1}_{B_\varepsilon(C_x)}(f(t_l t, x)) dt = 1.$$

Proof. For any $\tau > 0$ and any open set U , define an open set

$$N_\tau(x, U) = \{t: 0 \leq t < +\infty, f(\tau t, x) \in U\}.$$

It is easy to check that $\tau^{-1}N_1(x, U) = N_\tau(x, U)$. Then by Lemma 5.1 and Definition 1.1, it follows that $N_{t_1}(x, B_\varepsilon(C_x)), \dots, N_{t_l}(x, B_\varepsilon(C_x))$ all have the density 1. Thus

$$N_{t_1, \dots, t_l}(x, B_\varepsilon(C_x)) := N_{t_1}(x, B_\varepsilon(C_x)) \cap \cdots \cap N_{t_l}(x, B_\varepsilon(C_x))$$

also has the density 1 in \mathbb{R}_+ . This completes the proof of Lemma 5.2. \square

Recall that a motion $f(t, x)$ is said to be *Lagrange stable* as $t \rightarrow +\infty$ if the orbit-closure $\overline{O_f(x)}$ is compact in X (cf. [15]). As a direct result of Lemma 5.2, we can obtain the following.

Corollary 5.3. *For any Lagrange stable motion $f(t, x)$ as $t \rightarrow +\infty$, C_x is the minimal closed subset of X such that for any $t_1 > 0, \dots, t_l > 0$ and any $\varepsilon > 0$,*

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \mathbb{1}_{B_\varepsilon(C_x)}(f(t_1 t, x)) \cdots \mathbb{1}_{B_\varepsilon(C_x)}(f(t_l t, x)) dt = 1.$$

This result shows that C_x is the “minimal center of multi-attraction” of a Lagrange stable motion $f(t, x)$ as $t \rightarrow +\infty$.

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